Where mathematics comes from: How the embodied mind brings mathematics into being

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Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being

George Lakoff and Rafael E. Núñez

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[Reviewed by Bonnie Gold, on 02/3/2001]

Mathematical platonists, awake! Here, in a book written to put the final nail in the coffin of platonism, we have the beginnings of a response to the one serious philosophical challenge to platonism: how can human beings, finite physical beings, ever develop an understanding of mathematics, abstract and infinite, disjoint from physical experience?

George Lakoff, a linguist, and Rafael Núñez, a cognitive psychologist, have long-standing interests in mathematics. They have written a unique and fascinating, if flawed, book attempting to "apply the science of mind to human mathematical ideas" (p. *xi*) to discover where our mathematical ideas come from. This book introduces the discipline of "mathematical idea analysis:" how our understanding of human cognitive processes can account for the development of mathematical ideas.

Their thesis is that virtually all mathematical ideas arise as metaphors. *Virtually all*, because a bit of mathematics, called subitizing — the ability to recognize very small numbers — involves innate capacities of our brains. But subitizing cannot account even for arithmetic: while we can subitize numbers as large as 4, we can't subitize 4 + 3. For this we need metaphor. "Metaphor is not a mere embellishment; it is the basic means by which abstract thought is made possible. One of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts." (p. 39) Their metaphors, essentially isomorphisms, consist of a source domain, generally concrete; a target domain (the domain of the new objects being developed) and a mapping between the two. A very early example is Arithmetic As Object Collection (p. 54-55): the source domain is object collection, the target domain is arithmetic, collections of objects of the same size correspond to numbers, the size of the collection corresponds to the size of a number, the smallest possible collection to the number 1, etc.

Arithmetic As Object Collection is the simplest type of mathematical metaphor, a "grounding metaphor," using everyday experiences to ground abstract concepts such as addition. There are also "linking metaphors" which have both source and target domains within mathematics. An example is the metaphor Numbers Are Points on a Line (p. 279), with source domain points on a line, and target domain numbers; a point *P* corresponds to a number *P'*, the origin to the number 0; a designated unit distance point *I* to the number 1; etc. More sophisticated mathematical ideas may involve several metaphors at once, in a "conceptual blend." For example, Dedekind's Number-Line blend "uses two metaphors: Spaces Are Sets and Numbers are Points on a Line." (p. 295)

Their most important metaphor is the Basic Metaphor of Infinity (BMI). They start with finite but continuous iterative processes, involving a beginning state, intermediate states, and a final resultant state, and map them to iterative processes that go on and on, but the final resultant state now becomes actual infinity. (p. 159) This is no longer quite an isomorphism. Which is, in some ways, a good thing: their earlier metaphors which are simply isomorphisms seem rather sterile. But now that we're getting something genuinely new, the ambiguity of how to go from the intermediate states to the final state leaves a gap that needs more explanation than they give. In particular, it can lead to paradoxes such as those exposed by Zeno, or one they discuss that involves the limit of arc-lengths of semicircles with centers at $((2m-1)/2^n, 0)$, $m = 1, 2, ..., 2^{n-1}$: for each n, the semicircles' lengths add up to pi/2, yet the pointwise limit of the semicircles is the interval [0,1](p. 325 - 333).

Beginning with the BMI, they also start making rather frequent mathematical errors, particularly in their discussion of infinitesimals. For example, they develop the set of infinitesimals by constructing sets $S(n)$ which are "the set of all numbers greater than zero and less than 1/n, satisfying the first nine axioms for the real numbers" (p. 228). Among these axioms (p. 200) are "the existence of identity elements for both addition and multiplication" (axiom 4) and the existence of additive (axiom 5) and multiplicative (axiom 6) inverses. Unfortunately, there is *no* set of real numbers greater than 0 and less than 1 containing identities and inverses: their sets are inconsistent! This can be fixed, of course, by being a bit more detailed. But it gets worse when they attempt to develop the "granular numbers", which are basically the smallest subfield of the hyperreal numbers generated by the real numbers and one infinitesimal. They construct this field by attempting to pick out "the first infinitesimal number produced by the BMI" (sic, p. 235) — but there can be no *first* such number! (They believe the granular numbers are a new

mathematical object that they've discovered, and go on, p. 254, to discuss why mathematicians could have been so blind as not to have found them sooner!!)

The book deteriorates from the BMI onward. In the chapter on "Real Numbers and Limits", the authors observe, correctly, that our usual epsilon-delta definition of limit doesn't really capture how we conceptualize limits, the process of a function approaching a limit, partly because we allow all possible real numbers as values for epsilon, resulting in acceptable epsilons which are irrelevant to the limit. They decide, therefore, to use a sequential definition of limit. However, rather than following a standard treatment of this topic (involving all subsequences of the sequence under consideration), they introduce the concept of a sequence of "critical elements," which are "those terms of the sequence that must converge in order for the sequence as a whole to converge" (p. 195); their notion is incoherent. A bit later they confuse how the standard definition of limit works, believing they can choose epsilons as they please (p. 199); similar errors continue through much of the remaining mathematical content.

Not satisfied with introducing a new field of intellectual inquiry, the authors devote the penultimate part of the book (the last part is an extended explanation, in terms of their metaphors, of the mathematical basis of $e^{i\pi} + 1 = 0$) to the introduction of a new philosophy of mathematics which they believe to be implied by their conclusions. They assert that they have dealt a fatal blow to what they call the "Romance of Mathematics" (p. 339), roughly what is often referred to as platonism: "Mathematics is an objective feature of the universe ... What human beings believe about mathematics therefore has no effect on what mathematics really is. ... Since logic itself can be formalized as mathematical logic, mathematics characterizes the very nature of rationality. ..." As with many social constructivists (e.g., Reuben Hersh), they dislike this romance because "It intimidates people. ... It helps to maintain an elite and then justify it." (p. 341) Their arguments in favor of "human mathematics" are briefer and no more eloquent than those in Hersh's What is Mathematics, Really? and have little direct connection with the rest of the book. While I have no more sympathy than the authors for this elitism of mathematicians (and have devoted most of my life to undoing its effects), the elitism of mathematicians is no more a consequence of a belief that mathematical facts are an objective feature of the universe than the elitism of physicists is a consequence of their belief that physical facts are an objective feature of the universe. While their description of how humans develop concepts of mathematics is consistent with the restricted social constructivism of Hersh, it is also consistent with any reasonable version of platonism that distinguishes between mathematical facts and human knowledge of those mathematical facts. Indeed, as we begin to describe how human understanding of mathematical ideas is consonant with human understanding of other abstract systems, platonists will be able to respond to the challenge from philosophers of how finite corporeal beings can have contact with, and knowledge of, an infinite abstract branch of knowledge.

One small annoyance: the references are broken into 6 categories: to find "Narayanan [1997]," you may have to look through them all before finding the full reference.

Despite its flaws, this book is a significant contribution to our understanding of mathematics' relation to people. Although the analysis has some defects, most first attempts to introduce a new discipline involve some important insights but also some stumbling around in the dark. The insights these authors introduce make at least the first half of the book well worth reading for anyone (advanced undergraduate and up) interested in the philosophy of mathematics, or in the genesis of mathematical ideas.

The authors' reply to this review appears as one of the "reader reviews" below.

Bonnie Gold (bgold@monmouth.edu) is chair of the Mathematics Department at Monmouth University. Her interests include alternative pedagogies in undergraduate mathematics education and the philosophy of mathematics. She is editor of MAA Online's Innovative Teaching Exchange, and co-editor of Assessment Practices in Undergraduate Mathematics (MAA Notes #49). On the philosophical side, she is the author of "What is the Philosophy of Mathematics and What Should It Be" (*Mathematical Intelligencer*, 1994) and was the co-organizer of a session on the philosophy of mathematics at the January 2001 joint meetings in New Orleans.

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Comments

George Lakoff and Rafael E. Núñez
Submitted by akirak on Mon, 2013-01-28 10:58

With great interest we have read the review of our book by Bonnie Gold that appeared in the MAA online book review column. It is a serious and professional review and we are glad that she found our book "unique and fascinating" and "a significant contribution to our understanding to mathematics' relation to people." We also deeply appreciate her efforts to understand our work (which is not within the field of mathematics itself) and to try to describe it accurately to an audience of mathematicians. That is certainly not an easy task for someone outside the field of cognitive science. We think she has done a good job in trying to explain some of the main ideas of the "new discipline of intellectual inquiry" we introduce. We find, however, that important misunderstandings of the cognitive science occur in the review, misunderstandings referred to as "flaws," "mathematical errors," and "defects." We think that these are serious misunderstandings and they need to be clarified - all the more so because of the serious purpose and mathematical competence of the reviewer and amount of effort put into the review (all of which we praise highly).

First, there is a major misunderstanding from which many misinterpretations follow. We have the impression that Gold didn't really get the main thesis of our book: It is the human embodied mind that brings mathematics into being. This is precisely what the subtitle of the book explicitly indicates. "Embodiment" thus, with its strong biological and cognitive constraints, is a fundamental theoretical component that gives shape and continuity to

the entire book. It is under this view that the cognitive mechanisms we describe, among which conceptual metaphor is one of the most important ones, make sense at all: Conceptual metaphor has explanatory power precisely because they are empirically observable embodied mechanisms that satisfy strong biological and cognitive constraints. The term "embodiment" and its related concepts appear all over the book, to the point that both summarizing theoretical chapters at the end are entitled "The Theory of Embodied Mathematics" (Chapter 15) and "The Philosophy of Embodied Mathematics" (Chapter 16). In Gold's review, however, the term embodiment is not even mentioned once!

This is not just picky terminology. Understanding the role of embodiment and its biological and cognitive constraints is extremely important to get an idea of what our book is about. So when Gold refers to conceptual metaphors as being "essentially isomorphisms" she presents an inaccurate picture, which leads her to miss some fundamental points of the book, as well as the continuity and overall structure of it. First, isomorphisms are mathematical entities, while conceptual metaphors are not. Second, as mathematical entities within a mathematical subject matter, isomorphisms don't have to satisfy any scientific (and therefore empirical) constraints. Of course, within certain scientific applications, they can be applied to satisfy constraints, but they don't have to.

Conceptual metaphors, on the contrary, are empirically observable mechanisms of the human mind, which as explanatory constructs *must* satisfy strong biological and cognitive constraints. With a conceptual metaphor you have to explain data observed empirically, while with isomorphisms you don't necessarily have to. It is simply a confusion between disciplines to refer to conceptual metaphors as "isomorphisms." Our book is within the discipline of cognitive science and its subject matter is the cognitive science of mathematical ideas. To refer to conceptual metaphors as isomorphisms is to assume that the book is within the discipline of mathematics, which it is not. Our book is an attempt to give an account of mathematical ideas and inferences in terms of biologically and cognitively plausible mechanisms of the human mind, such as conceptual metaphors.

For example, the conceptual metaphor Arithmetic As Object Collection to which Gold refers is not a mere descriptive isomorphism. It is the embodied cognitive mechanism that gives an account of why the empirically observed expressions that exist in human (even technical) communication such as "three is *bigger* than two" or "four is *smaller* than eight" have the precise meaning they have, despite the fact that numbers in themselves don't have size. We believe that, because Gold misses the deep role of embodiment in our theoretical account throughout the book, she sees our characterization of "human mathematics" as "brief" and "not eloquent" enough, and having "little direct connection with the rest of the book."

We would also like to clarify a couple of technical details regarding how conceptual metaphors work. Gold says, correctly, that the Basic Metaphor of Infinity (BMI) is the most important metaphor in the book, and she points out (also correctly) that the BMI is not an isomorphism (that's right!). Gold accurately observes that the Basic Metaphor of Infinity characterizes "something genuinely new" (i.e., an *end* to an *un*ending process: actual infinity). Unfortunately, she seems not to understand what conceptual metaphors are and how they differ in kind from disembodied mathematical isomorphisms (which are literal, not metaphorical). As a result, she mistakenly claims that the BMI introduces an "ambiguity of how to go from the intermediate states to the final state", leaving "a gap that needs more explanation." It is incorrectly taking conceptual metaphors to be mathematical isomorphisms that generate that gap. Conceptual metaphors, being human cognitive mechanisms have many properties not captured by isomorphisms. As we say it explicitly in pages 45 and 46, "conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also *introduce new elements* into the target domain" (italics in the book). These elements are not inherent to the target domain. In the BMI case, an *end* is not inherent to the target domain constituted by an *unending* process that goes on and on. It is the BMI that brings forth this new metaphorical entity: an ending to an unending process. Asking the question of how (exactly) to go from the intermediate states to the final state is a question that belongs to the realm of literal, not metaphorical processes. We give a simple example on page 46.

The moral here is this: It is totally consistent with what we know about human cognitive mechanisms that actual infinity could be a metaphorical idea. Via a specific conceptual metaphor (the BMI), an unending iterative process that goes on and on can be conceptualized as a process with an actual end and an actual final resultant state (which are precisely the elements not inherent to the target domain of unending processes).

Now, regarding the "mathematical errors" mentioned by Gold, she is in some extent right. There are some errors in the text of the first printing. But there are several things to say about them. First, several of these mistakes are editorial errors (not ours) that unfortunately affect mathematical content, which may mislead a careful reader, especially those who are mathematically trained (e.g., the definition of limits in p. 199). These editorial mistakes (and others) have been corrected in the errata section of the website for our book, which we set up soon after the publication of the book (www.unifr.ch/perso/nunezr). Second, there are indeed some passages in which we stated things in a sloppy way and we apologize (e.g., the characterization of the infinitesimals through the BMI in page 228, correctly pointed out by Gold). We agree with Gold, that these passages "can be fixed, of course, by being a bit more detailed." We will fix them in the next printing and place a corrected version on our website as soon as possible. Some corrections are already there. In any case, to our knowledge, none of these errors affect the substance of our arguments.

Finally, there is a whole group of concepts and analyses we present that are incorrectly called "mathematical errors" because they are taken as mathematical analyses and not as cognitive analyses. We think that, because of this important misunderstanding, Gold has made the judgment that "the book deteriorates from the BMI onward." For example she says that we "decide" to use a sequential definition of limit in which we don't "follow a standard treatment of this topic." The problem is that the kind of standard treatment Gold refers to is mathematical in nature, not cognitive. Our job is not to improve mathematics (how could we possibly do that?). Nor is it our job to duplicate what occurs in standard math texts (that isn't cognitive science!). And it is certainly not our job to tell mathematicians how to do mathematics (just as it not the job of a zoologist to tell a bird how to fly!). As cognitive scientists our job is to give cognitive accounts of largely unconscious mechanisms of mind used to characterize ideas - ideas of all sorts, both mathematical and otherwise.

Therefore, when it comes to limits, we ask different questions than mathematicians do. It is part of our job to understand ideas such as "the limit of f(x) as x *approaches* a." As cognitive scientists, we have to answer questions like: From a cognitive perspective, what is approaching what? What is moving? From where to where? In what (cognitive) space? And so on. These are questions about a human understanding of a humanly created idea: *approaching* a limit. This is inherently a dynamic idea, not captured by the standard mathematical treatment using a static logical expression with epsilons and deltas. Thus, characterizing limits in terms of dynamic sequences is not a "decision" we make; rather it is the reality of how people think, a cognitive reality that we must explain in cognitively plausible terms.

Gold's misunderstanding of our goals and intentions can be more clearly seen through her interpretation of what we have called "granular numbers." Nowhere in our book do we say that "the granular numbers are a new mathematical object that we have discovered" (how could we possibly make such a statement!). To begin with, "discovering" such a thing would be inconsistent with the non-platonic nature of embodied mathematics we endorse. What we say instead is that we have "invented the granulars by applying the BMI" (p. 254). This is overtly *not* a process within formal mathematics. It is the use of a cognitive process for creating mathematical ideas. In this case, we use everyday cognitive mechanisms such as the BMI together with the inferential structure of the idea of "speck" (which we hypothesize as the everyday idea that was the inspiration for Leibniz's idea of infinitesimal).

It is in the realm of this cognitive exercise that we can affirm that the BMI produces "the first infinitesimal" (p. 235). This is not a mathematical result in the classic sense (i.e., it is not a result obtained by proving a mathematical theorem). Therefore it is not a "mathematical error" to say such a thing. "The first infinitesimal" is a consequence of the inferential structure of the BMI when applied to the particular case being discussed. The fact that this infinitesimal is the "first infinitesimal," is an entailment of the metaphor, which generates a unique final resultant state, with no prior resultant state of that kind. That is why it is "first." For all these reasons, Gold is incorrect when she says on page 254 that we "discuss why mathematicians could have been so blind as not to have found them [the granulars] sooner." In that passage we ask an entirely different question: Why didn't the remarkable mathematicians who have worked on the hyperreals develop such a number system when they could have done it easily? On the same page we give our answer, which doesn't have anything to do with considering these mathematicians "so blind as not to have found them [the granulars] sooner." That view simply does not fit with the respect bordering on awe that we have for those extraordinary mathematicians.

We hope these clarifications help mathematicians to understand (and to enjoy!) our book better. We are aware that it is not a simple task to try to follow a cognitive analysis of mathematical ideas when one has been trained as a professional mathematician, which is probably the case of most readers of this MAA column. We are also aware, and here we agree with Gold, that "most first attempts to introduce a new discipline involve some important insights but also some stumbling in the dark." So, it is true that we are just starting this enterprise, and therefore there are still many unclear components. We see the cognitive science of mathematics as a multidisciplinary field, and we certainly need all the help we can get from professional mathematicians. We have to keep in mind, however, that our goal is to characterize mathematics in terms of cognitive mechanisms, not in terms of mathematics itself, e.g., formal definitions, axioms, and so on. Indeed, part of our job is to characterize how such formal definitions and axioms are themselves understood in embodied cognitive terms.

We simply have a different job than professional mathematicians have. We have to answer such questions as: How can a number express a concept? How can mathematical formulas and equations express general ideas that occur outside of mathematics, ideas like recurrence, change, proportions, self-regulating processes, and so on? How do ideas within mathematics differ from similar (but not identical) ideas outside mathematics (e.g., the idea of "space" or "continuity")? How can "abstract" mathematics be understood? What cognitive mechanisms are used in mathematical understanding?

We hope that such questions asked and answered from outside of mathematics proper will interest mathematicians. And we hope that they will not be mistaken for questions and answers within mathematics.

Finally, we want to thank Bonnie Gold once more for her review, and to thank the MAA online book review column for finding a reviewer of such mathematical competence and with the openness and energy required for such an undertaking. We have the greatest respect for her. We are all too aware of how much effort goes into such a review and how misunderstandings can arise naturally across disciplines. It is only through a forum such as this that such issues can be aired in a spirit of cooperation and honest inquiry.

George Lakoff and Rafael E. Núñez

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How the Embodied Mind Brings Mathematics Into Being Author: George Lakoff,Rafael E. Núñez Publisher: Basic Books (AZ) ISBN: N.A Category: Mathematics Page: 493 View: 9502. Continue Reading →. Provides an in-depth analysis of the cognitive science of mathematical ideas that argues that conceptual metaphor plays a definitive role in mathematical ideas, exploring such concepts as arithmetic, algebra, sets, logic, and infinity. What underlies numerical and mathematical disabilities and disorders, and what is the prognosis for rehabilitation? These questions are the domain of mathematical cognition, the field of research concerned with the cognitive and neurological processes that underlie numerical and mathematical abilities. George Lakoff, Rafael NuÃ±ez, "Where Mathematics Come From: How The Embodied Mind Brings Mathematics Into Being" 2001 | ISBN-10: 0465037712 | 512 pages | PDF | 26 MB. This book is about mathematical ideas, about what mathematics means-and why. Abstract ideas, for the most part, arise via conceptual metaphor-metaphorical ideas projecting from the way we function in the everyday physical world. Where Mathematics Comes From argues that conceptual metaphor plays a central role in mathematical ideas within the cognitive unconscious-from arithmetic and algebra to sets and logic to infinity Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being (hereinafter WMCF) is a book by George Lakoff, a cognitive linguist, and Rafael E. Núñez, a psychologist. Published in 2000, WMCF seeks to found a cognitive science of mathematics, a theory of embodied mathematics based on conceptual metaphor. Mathematics makes up that part of the human conceptual system that is special in the following way